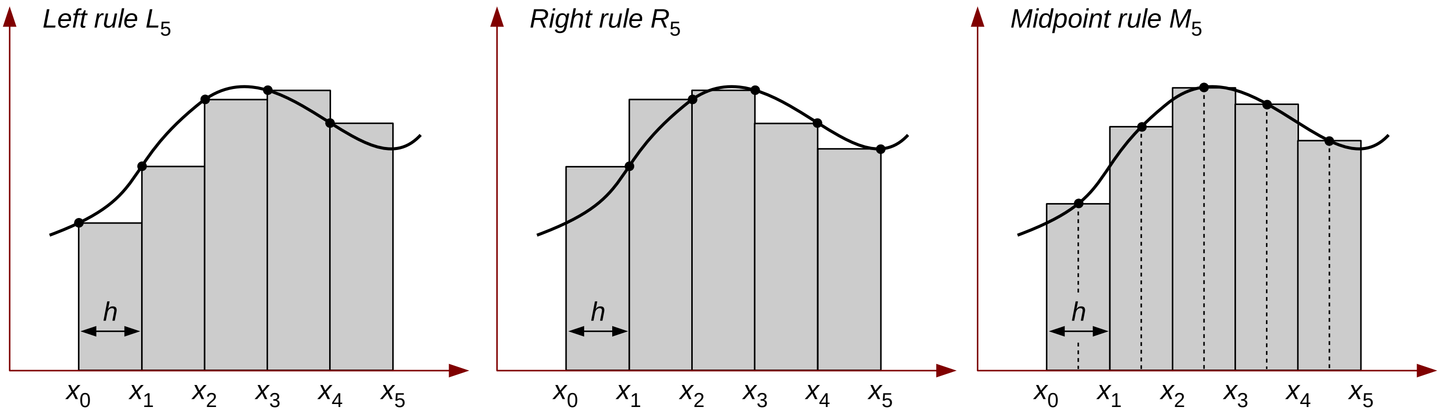
Numeric integration

Left, Right and Midpoint Rules

Divide the region of interest into n equally-spaced sections, and approximate the area under the curve with a rectangle in each section. We can align the rectangles to the curve on the left, on the right, or in the middle.



To approximate the integral $\int_a^b f(x)\,dx$ using n equal-spaces sections, we set the width of each rectangle to:

$$h = \frac{b-a}{n}$$

and set x-values at:

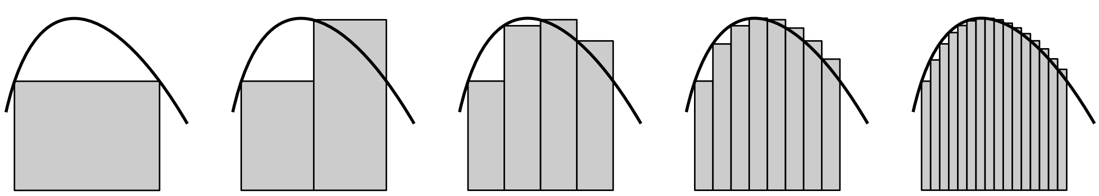
$$\begin{aligned} x_0 &= a \\ x_n &= b \\ x_k &= x_0 + k \times h \ \forall\ k \in \{0 \ldots n\} \end{aligned}$$

Left rule: $$L_n = h\left[f(x_0) + f(x_1) + f(x_2) + \ldots + f(x_{n-1})\right] = h\sum_{k=0}^{n-1} f(x_k)$$

Right rule: $$R_n = h\left[f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + f(x_n)\right] = h\sum_{k=1}^{n} f(x_k)$$

Midpoint rule: $$M_n = h\left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \ldots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right] = h\sum_{k=0}^{n-1} f\left(\frac{x_k+x_{k+1}}{2}\right)$$

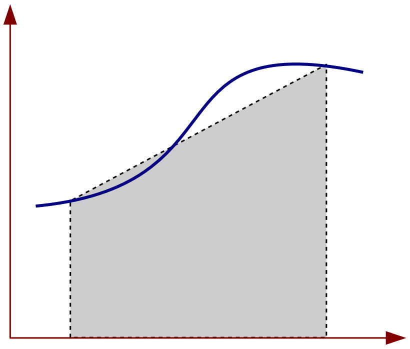
In each case, as n increases, the rectangles match the curve more closely and the approximation becomes better:



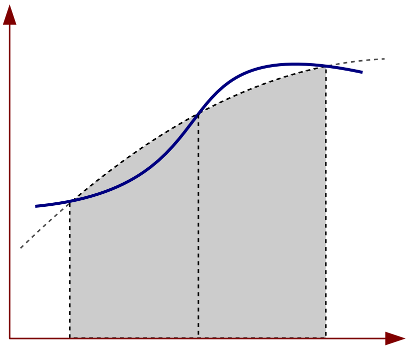
Numeric integration

Trapezoidal and Simpson's Rules

For the Trapezoidal Rule, we use trapezoids (U.S. English) or trapeziums (British English) rather than rectangles. For Simpson's Rule, we approximate each section of the curve with a parabola.



Trapezoidal Rule: the curve is approximated by a straight line, the area formed is a trapezoid (US English) or trapezium (UK English).



Simpson's Rule: each section of the curve is approximated by a parabola passing through the two ends and the midpoint.

Like the Left, Right and Midpoint Rules, we approximate the integral by dividing the region of interest into n equally-spaced sections. As n increases, the approximation gets better.

Trapezoid rule:

$$T_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$
$$= \frac{h}{2}\left[f(x_0) + 2\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right]$$

The Trapezoid Rule is equal to the average of the Left and Right Rules: $T_n = \dfrac{L_n + R_n}{2}$

Simpson's rule:

$$S_n = \frac{h}{3}\left[f(x_0) + 4\sum_{k=1}^{n-1} f(x_{2k-1}) + 2\sum_{k=1}^{n-2} f(x_{2k}) + f(x_n)\right] = \frac{2M_n + T_n}{3}$$

There is a version of Simpson's Rule, known as “Simpson's 3/8 Rule”, that interpolates the curve with a cubic rather than a parabola:

Simpson's 3/8 rule:

$$S_n = \frac{3h}{8}\left[f(x_0) + 3\sum_{k=1,4,\ldots}^{n-2} f(x_k) + 3\sum_{k=2,5,\ldots}^{n-1} f(x_k) + 2\sum_{k=3,6,\ldots}^{n-3} f(x_{2k}) + f(x_n)\right]$$

For Simpson's Rule, we must divide the region into an even number of points: $n = 2m$.
For Simpson's 3/8 Rule, we must divide the region into a multiple of three points: $n = 3m$.

Boole's Rule

Boole's Rule (sometimes misspelled Bode's Rule) is another method of approximating an integral. It uses five equally-spaced points with $x_1 = a$ and $x_5 = b$:

$$\int_a^b f(x)\,dx \approx \frac{2h}{45}\left[7f(x_1) + 32f(x_2) + 12f(x_3) + 32f(x_4) + 7f(x_5)\right]$$